

$$1. \frac{\sqrt{10}}{10}$$

$$2. \frac{5}{4}$$

$$3. -\frac{3}{5}, \frac{\sqrt{2}}{10}.$$

$$4. \text{解: (1): } \because \cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{10}, \text{ 又 } x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

$$\therefore x - \frac{\pi}{4} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\therefore \sin\left(x - \frac{\pi}{4}\right) = \frac{7\sqrt{2}}{10}$$

$$\therefore \sin x = \sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \sin\left(x - \frac{\pi}{4}\right) \cos \frac{\pi}{4} + \cos\left(x - \frac{\pi}{4}\right) \sin \frac{\pi}{4}$$

$$= \frac{7\sqrt{2}}{10} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{10} \times \frac{\sqrt{2}}{2} = \frac{4}{5}$$

$$(2): \text{由 (1) 知 } \sin x = \frac{4}{5}, \quad x \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

$$\therefore \cos x = -\frac{3}{5}$$

$$\therefore \sin 2x = -\frac{24}{25}, \cos 2x = -\frac{7}{25}$$

$$\begin{aligned} \therefore \sin\left(2x + \frac{\pi}{3}\right) &= \sin 2x \cos \frac{\pi}{3} + \cos 2x \sin \frac{\pi}{3} \\ &= \left(-\frac{24}{25}\right) \times \frac{1}{2} - \frac{7}{25} \times \frac{\sqrt{3}}{2} = -\frac{24 + 7\sqrt{3}}{50} \end{aligned}$$

$$5. \text{解由题知 } \cos \alpha = \frac{3}{5}, \sin \beta = \frac{12}{13}, \text{ 因为锐角 } \alpha \text{ 和钝角 } \beta$$

$$\text{所以 } \sin \alpha = \frac{4}{5}, \cos \beta = -\frac{5}{13}$$

$$\text{所以 } \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{33}{65}.$$